A FIXED POINT THEOREM FOR A SELFMAP OF A COMPACT D*-METRIC SPACE

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ABSTRACT

The purpose of this paper is to prove a fixed-point theorem for selfmap of a compact D-metric space, and we show that a fixed point theorem of metric space proved by Brain Fisher ([5], Theorem 2) follows as a particular case of our theorem*

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Key words: *D*-metric space, Fixed point theorem*

INTRODUCTION

The study of metric fixed point theorem has been researched extensively in the past decades, since fixed point theory plays a major role in Mathematics and Applied Sciences, such as Optimization, Mathematical Economics, Theory of Differential Equations, Mathematical Models and Potential Theory.

Different mathematicians tried to generalize the usual notion of metric space (X, d). In 1992 Dhage [2] has initiated the study of generalized metric space called D- metric space and fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately, almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [6], [7] and [8]. Recently Shaban Sedghi, Nabi Shobe and Haiyun Zhou [9], have introduced D*- metric spaces as a probable modification of D- metric spaces and proved some fixed point theorems.

In this paper, using the concept of D*- metric space we prove a fixed point theorem. Our theorem generalizes the Brain Fisher [5] fixed point theorem of metric space. Before starting and proving our results, we shall recall some mathematical preliminaries.

MATHEMATICAL PRELIMINARIES

Definition 2.1([9]): Let X be a non-empty set. A function $D^*: X^3 \to [0, \infty)$ is said to be a **generalized metric** or **D*-metric** or **G-metric** on X, if it satisfies the following conditions

i. $D^*(x, y, z) \geq 0$ for all $x, y, z \in X$.
ii. $D^*(x, y, z) = 0$ if and only if $x = y = z$.

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iii. $D^*(x, y, z) = D^*(\sigma(x, y, z))$ for all $x, y, z \in X$
Where $\sigma(x, y, z)$ is any permutation of the set $\{x, y, z\}$.

iv. $D^*(x, y, z) \le D^*(x, y, w) + D^*(w, z, z)$ for all $x, y, z, w \in X$.

The pair (X, D^*), where D^* is a generalized metric on X is called a

D*–metric space or a **generalized metric space**.

Example 2.1: Let (X, d) be a metric space. Define $D_1^*: X^3 \to [0, \infty)$ by
$D_1^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for $x, y, z \in X$. Then (X, D_1^*) is a generalized metric space.

Example 2.2: Let (X, d) be a metric space. Define $D_2^*: X^3 \to [0, \infty)$ by
$D_2^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for $x, y, z \in X$. Then (X, D_2^*) is a generalized metric space.

Example 2.3: Let $X = R$, **define** $D^*: R^3 \to [0, \infty)$ by 0
if $x = y = z$
$D^*(x, y, z) =$
Max $\{x, y, z\}$ otherwise.
Then (R, D^*) is a generalized metric space.

Note 2.4: Using the inequality in (iv) and (ii) of Definition 2.1, one can prove that if (X, D^*) is a D*–metric space, then
$D^*(x, x, y) = D^*(x, y, y)$ for all $x, y, \in X$.
Infact $D^*(x, x, y) \le D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and
$D^*(y, y, x) \le D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$,
proving the inequity.

Definition2.5: Let (X, D*) be a D*-metric space. For $x \in X$ and $r > 0$, the set
$B_{D^*}(x, r) = \{y \in X; D^*(x, y, y) < r\}$ is called the **open ball** of radius r about x.

For example, if $X = R$ and $D^*: R^3 \to [0, \infty)$ is defined by
$D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all $x, y, z \in R$. Then

$B_{D^*}(0, 1) = \{y \in R; D^*(0, y, y) < 1\}$

$= \{y \in R; 2|y| < 1\}$

$= \{y \in R; |y| < ½\} = (-½, ½)$.

Definition 2.6: Let (X, D^*) be a D^*–metric space and $E \subset X$.

(i) If for every $x \in E$, there is a $\delta > 0$ such that $B_{D^*}(x, \delta) \subset E$, then E is said to be an **open subset** of X

(ii) If there is a $k > 0$ such that $D^*(x, y, y) < k$ for all $x, y \in E$ then E is said to be ***D*–bounded.***

It has been observed in [9] that, if τ is the set of all open sets in (X, D^*), then τ is a topology on X (called the **topology induced by the *D**–metric**) and also proved that $B_{D^*}(x, r)$ is an open set for each $x \in X$ and $r > 0$ ([9], Lemma 1.5). If (X, τ) is a compact topological space we shall call (X, D^*) is a **compact** D^*–metric space.

Definition 2.7: Let (X, D^*) be a D^*–metric space. A sequence $\{x_n\}$ in X is said to

(i) **converge to *x*** if $D^*(x_n, x_n, x) = D^*(x_n, x, x) \to 0$ as $n \to \infty$

(ii) be a **Cauchy sequence**, if to each $\in > 0$, there is a natural number n_0 such that $D^*(x_n, x_n, x_m) < \in$ for all $m, n \geq n_0$.

It is easy to see (infact proved in [9], Lemma 1.8 and Lemma 1.9) that, if $\{x_n\}$ converges to x in (X, D^*) then x is unique and that $\{x_n\}$ is a Cauchy sequence in (X, D^*). However, a Cauchy sequence in a (X, D^*) need not be convergent as shown in the example given below.

Example 2.8: Let $X = (0, 1]$ and $D^*(x, y, z) = | x - y | + | y - z | + | z - x |$ for $x, y, z \in X$, so that (X, D^*) is a D^*–metric space.

Define $x_n = \frac{1}{n}$ for $n = 1, 2, 3 \ldots\ldots,$ then $D^*(x_n, x_n, x_m) = 2 | x_n - x_m | = 2 | \frac{1}{n} - \frac{1}{m} |$, so that

$D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$, proving $\{x_n\}$ is a Cauchy sequence in (X, D^*). Clearly $\{x_n\}$ does not converge to any point in X.

Definition 2.9: *A* D^*–metric space (X, D^*) is said to **complete** if every Cauchy sequence in it converges to some point in it.

It follows that the D^*–metric space given in Example 2.8 is not complete.

Note 2.10: We have seen (In Example 2.1 and Example 2.2) that on any metric space (X, d) it is possible to define at least two D^*–metrics, namely D_1^* and D_2^*, using the metric d. We shall call D_1^* and D_2^* as D^*–metrics induced by d. Thus every metric space (X, d) gives rise to at least two D^*–metric spaces (X, D_1^*) and (X, D_2^*). Also if (X, D^*) is a D^*–metric then defining $d_0(x, y) = D^*(x, y, y)$ for $x, y \in X$, we can show easily that (X, d_0) is a metric space and we shall call d_0 as a metric induced by D^*.

The following result is of use for our discussion.

Theorem2.11: Let (X, d) be a metric space and $D_i^*(i = 1, 2)$ be the two D^*– metrics induced by d (given in Example 2.1 and Example 2.2). For any i (=1, 2) a sequence $\{x_n\}$ in (X, D_i^*)

is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).

Proof: - First note that for i =1, 2 we have
$d(x, y) \leq D_i^*(x, y, y) \leq 2d(x, y)$ for all $x, y, \in X$.
Now the theorem follows immediately in view of the above inequality.

For example, if $\{x_n\}$ is a Cauchy sequence in (X, d), then for any given $\epsilon > 0$ choose a natural number n_0 such that $m, n \geq n_0$ implies $d(x_m, x_n) < \frac{\epsilon}{2}$; and note that for the same n_0 we have
$m, n \geq n_0$ implies $D_i^*(x_m, x_n, x_n) \leq 2d(x_m\ x_n) < \epsilon$,
proving that $\{x_n\}$ is a Cauchy sequence in (X, D_i^*).

Similarly, the other part of the theorem can be proved using the other inequality noted in the beginning of the proof.

Corollary 2.12: Suppose (X, d) is a metric space. Let D_1^* and D_2^* be two D^*– metrics induced by d, then for any i (=1, 2) the space (X, D_i^*) is complete if and only if (X, d) is complete.

Proof: - Follows from Theorem 2.11.

Definition 2.13:- If (X,D^*) is a D^*-metric space, then D^* is a **continuous function** on X^3, in the sense that $\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$, whenever $\{(x_n, y_n, z_n)\}$ in X^3 converges to $(x, y, z) \in X^3$. Equivalently, $\lim_{n\to\infty} x_n = x$, $\lim_{n\to\infty} y_n = y$, $\lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$.

THE MAIN RESULT

Theorem 3.1: If T is a selfmap of a D^*-metric space (X, D^*) into itself such that

(i) there is a $z \in X$, with $D^*(z, Tz, Tz) \leq D^*(x, Tx, Tx)$ for all $x \in X$,

and

(ii) $D^*(Tx, Ty, Ty) < \frac{1}{2}[D^*(x, Ty, Ty) + D^*(y, Tx, Tx)]$ for all $x, y \in X$, with $x \neq y$.

Then T has a unique fixed point.

Proof: Suppose $T: X \to X$ is such that $f(z) = D^*(z, Tz, Tz) \leq D^*(x, Tx, Tx) = f(x)$ for all $x \in X$.
That is, $f(z) \leq f(x)$ for all $x \in X$. Now we claim that $Tz = z$.
If $Tz \neq z$, then by (ii) we have

$$D^*(Tz, T^2z, T^2z) < \frac{1}{2}[D^*(z, T^2z, T^2z) + D^*(Tz, Tz, Tz)]$$
$$= \frac{1}{2} D^*(z, T^2z, T^2z)$$

$$< \frac{1}{2}[D^*(z, Tz, Tz) + D^*(Tz, T^2z, T^2z)],$$

which implies $D^*(Tz, T^2z, T^2z) < D^*(z, Tz, Tz)$. That is, $f(Tz) < f(z)$, a contradiction to the definition of f(z). Hence Tz = z, giving z is a fixed point of T.

Now suppose that T has another fixed point, say z' with $z \neq z'$, then by (ii) we have

$D^*(z, z', z') = D^*(Tz, Tz', Tz') < \frac{1}{2}[D^*(z, Tz', Tz') + D^*(z', Tz, Tz)]$,

which implies $D^*(z,z',z') < D^*(z,z',z')$, a contradiction. Therefore, we have z=z'. That is z is the unique fixed point of T.

As a consequence, we have a fixed point theorem for a selfmap of a compact D*-metric space.

Corollary 3.2: If T is a selfmap of a compact D*-metric space (X, D*) such that $D^*(Tx, Ty, Ty) \leq \frac{1}{2}[D^*(x, Ty, Ty) + D^*(y, Tx, Tx)]$ for all $x, y \in X$, with $x \neq y$. Then T has a unique fixed point.

Proof: Writing $f(x) = D^*(x, Tx, Tx)$ for $x \in X$, it can be seen that f is a continuous real function on X. Since X is compact, f attains minimum at some $z \in X$. That is $f(z) \leq f(x)$ for all $x \in X$. That is, $z \in X$ is such that $D^*(z, Tz, Tz) \leq D^*(x, Tx, Tx)$ for all $x \in X$, which is the condition (i) of Theorem 3.1. Also, by the hypothesis, condition (ii) of Theorem 3.1 holds. Hence the corollary follows from Theorem 3.1.
Now we deduce the following theorem

Corollary3.3 ([5] Theorem 2): If T is a selfmap of a metric space (X, d) into itself such that

i. T is continuous
ii. X is compact, and
iii. $d(Tx, Ty) < \frac{1}{2}[d(x, Ty) + d(y, Tx)]$ for all $x, y \in X$, with $x \neq y$,

then T has a unique fixed point.

Proof: Given (X, d) is a metric space. Then
$D_1^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$, then (X, D_1^*) is a D*-metric space on X and $D_1^*(x, y, x) = d(x, y)$. Therefore, condition (iii) gives

$$D_1^*(Tx, Ty, Tx) < \frac{1}{2}[D_1^*(x, Ty, x) + D_1^*(y, Tx, y)]$$
$$= \frac{1}{2}[D_1^*(x, Ty, Ty) + D_1^*(y, Tx, Tx)]$$

Which is the same as condition of Corollary 3.2. Also, since (X, d) is complete, we have that (X, D_1^*) is complete, hence the corollary follows.

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